MATEMATIKA 6F

27.2.2006

$X: \Omega \to \mathbb{R}$

$P[X \in B]$ B… Borelovy množiny – intervaly, spočetné sjednocení intervalů

Pravděpodobnostní míra $P[X \in \mathbb{R}] = 1$

$P[X \in \bigcup I_n] = \sum_{n=1}^{\infty} P[X \in I_n]$, I_n jsou disjunktní

$P[X \in (-\infty, t)] = \underbrace{P[X < t]}$ pro $\forall t \in \mathbb{R}$

$\to F_X(t)$ – distribuční funkce → spojitá z leva

$P[X \in (a,b)] = P[X<b] - \underbrace{P[X \leq a]}_{P[X<a]+P[X=a]} = F(b) - \lim_{t \to a+} F(t)$

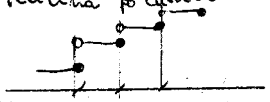
$P[X = c] = \lim_{t \to c+} F(t) - F(c)$ $\quad P[X<a] + P[X=a] = F(a) + \lim_{t \to a+} F(t) - F(a)$

$P[X \in \langle a,b)] = F(b) - F(a)$

$P[X \in (a,b\rangle] =$

$P[X \in (a,+\infty)] = 1 - \ldots$

Veličina po částech konstantní

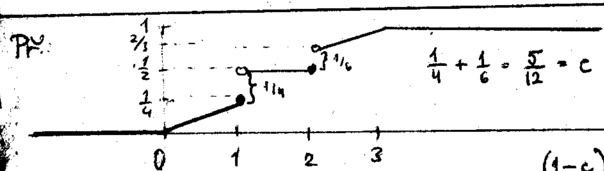


Hustota f · $F(t) = \int_{-\infty}^{t} f(x)\,dx$

Smíšená pravděpodobnost $F_Z(t) = c\,\underset{\text{diskrétní}}{F_X(t)} + (1-c) \cdot \underset{\text{spojitá}}{F_Y(t)}$

$Z = MIX_c(X,Y)$ $\quad P[Z \in B] = c \cdot P[X \in B] + (1-c)\,P[Y \in B]$

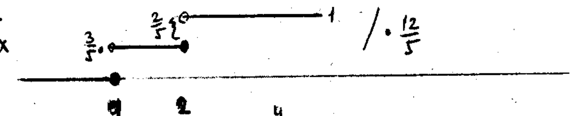
Př:



$\frac{1}{4} + \frac{1}{6} = \frac{5}{12} = c$

$(1-c) = \frac{7}{12}$

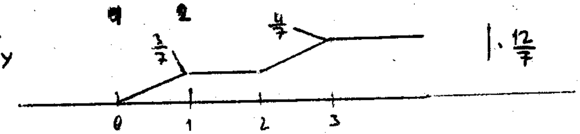
F_X



$/ \cdot \frac{12}{5}$

$F_X(t) = \begin{cases} (-\infty, 1\rangle : 0 \\ (1, 2\rangle : \frac{3}{5} \\ (2, \infty) : 1 \end{cases}$

F_Y



$| \cdot \frac{12}{7}$

F $\quad F_Y(t) = \begin{cases} t \in (-\infty, 0) : 0 \\ t \in (0,1\rangle : \frac{3}{7}t \\ t \in (1,2\rangle : 3/7 \\ t \in (2,3\rangle : \frac{1}{7}(t-2) + \frac{3}{7} \\ t \in (3, +\infty) : 1 \end{cases}$

$Y = h(X)$ $\quad X: \Omega \to \mathbb{R}$, $Y: \Omega \to \mathbb{R}$, $Y = X \circ h$

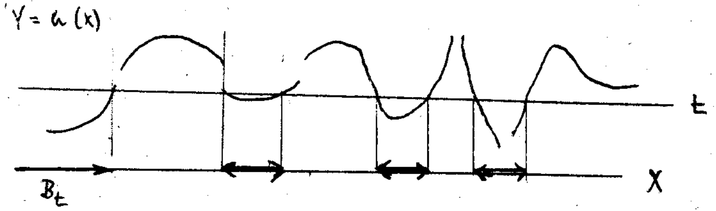
$Y = X + c$: $F_{X+c}(t) = P[X + c < t] = P[X < t - c] = F_X(t - c)$

$a > 0$: $Y = a \cdot X$ $\quad F_{a\cdot X}(t) = P[a \cdot X < t] = P[X < \frac{t}{a}] = F_X(\frac{t}{a})$

$a < 0$: $F_{aX}(t) = P[aX < t] = P[X > \frac{t}{a}] = 1 - \underbrace{P[X \leq \frac{t}{a}]}_{P[X<\frac{t}{a}] + P[X=\frac{t}{a}]} = 1 - P[X < \frac{t}{a}] - P[X = \frac{t}{a}]$

$= 1 - F_X(\frac{t}{a}) - \left(\lim_{t \to a+} F(\frac{t}{a}) - F(\frac{t}{a})\right)$

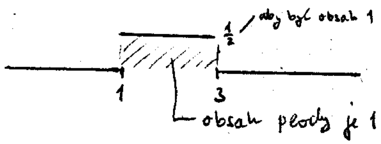
$= 1 - \lim_{t \to a+} F(\frac{t}{a})$



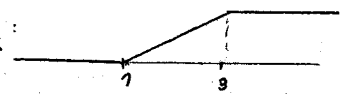
$F_{h(X)}(t) = P[h(X) < t] = P[X \in B_t]$

Př.:

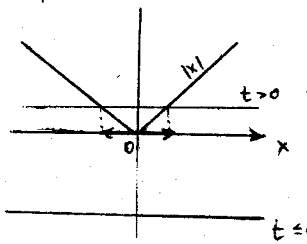
$X: f$



F_X:



$Y = |X|$



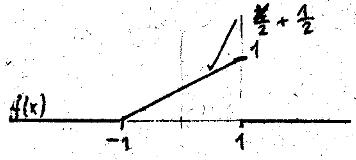
$F_{|X|}(t) = P[|X| < t] = \begin{cases} t \in (-\infty, 0\rangle : 0 \\ t > 0 : P[X \in \langle -t, t)] = \begin{cases} t < 1 : 0 \\ t \in \langle 1, 3\rangle : \int_1^t f(x)\,dx = \frac{t-1}{2} \\ t > 3 : 1 \end{cases} \end{cases}$

7.6.2006

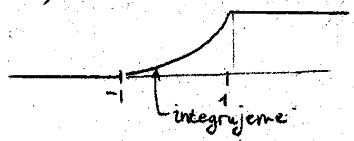
10.4.2.:

$X \div f(x)$

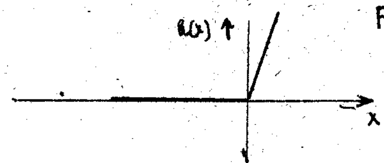
$Y = |X| + X$



Rozdělení Y, známe rozdělení X.



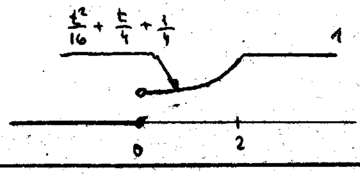
$h: \mathbb{R} \to \mathbb{R}$

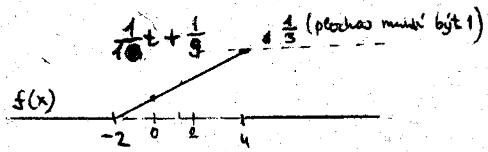


$F_Y(t) = P[Y < t] = P[\overbrace{|X| + X}^{h(x)} < t] = \begin{cases} t \leq 0 : 0 \\ t > 0 : P[X \in (-\infty, \frac{t}{2})] = \begin{cases} \frac{t}{2} < 1 : \int_{-1}^{t/2} (\frac{x}{2} + \frac{1}{2})\,dx = \dots \\ \frac{t}{2} > 1 : 1 \end{cases} \end{cases}$

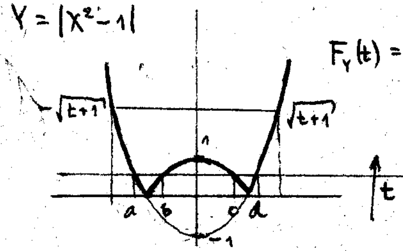
$\int_{-1}^{t/2} (\frac{x}{2} + \frac{1}{2})\,dx = \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-1}^{t/2} =$

$= \frac{1}{2}\left(\frac{t^2}{8} + \frac{t}{2} - \left[\frac{1}{2} - 1\right]\right) = \frac{t^2}{16} + \frac{t}{4} + \frac{1}{4}$





$Y = |X^2 - 1|$



$a = -\sqrt{t+1}$
$b = -\sqrt{1-t}$
$c = \sqrt{1-t}$
$d = \sqrt{t+1}$

$F_Y(t) = P[Y < t] = P[|X^2 - 1| < t] = \begin{cases} t \leq 0 : 0 \\ t \in (0, 1\rangle : P[X \in (a,b) \cup (c,d)] = * \\ t > 1 : P[X \in (-\sqrt{t+1}, \sqrt{t+1})] = ** \end{cases}$

$* = \int_{-\sqrt{t+1}}^{-\sqrt{1-t}} \frac{1}{18}x + \frac{1}{9}\,dx + \int_{\sqrt{1-t}}^{\sqrt{t+1}} \frac{1}{18}x + \frac{1}{9}\,dx$

$** = \begin{cases} -\sqrt{t+1} > -2 \dots t < 3, \quad t \in (1, 3), \quad \int_{-\sqrt{t+1}}^{\sqrt{t+1}} \frac{1}{18}x + \frac{1}{9}\,dx \\ \sqrt{t+1} < 4 \dots t < 15, \quad t \in \langle 3, 15) \quad \int_{-2}^{\sqrt{t+1}} \frac{1}{18}x + \frac{1}{9}\,dx \\ t \geq 15 : 1 \end{cases}$

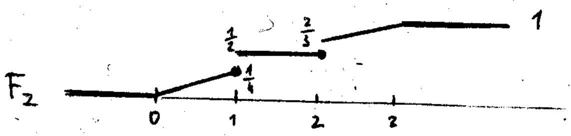
pozor na f(x)

KVANTILOVÁ FUNKCE NÁHODNÉ VELIČINY

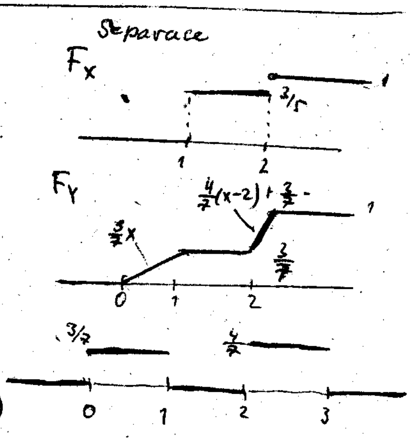
$EX = \int_{-\infty}^{\infty} x f(x)\,dx$ $\qquad EX = \int_0^1 Q(\alpha)\,d\alpha$

$EX = \sum_{x_i} x_i P[X = x_i]$

$EZ = c \cdot EX + (1-c) \cdot EY$



Separace



$F_z = c \cdot F_X + (1-c) F_Y = \frac{5}{12} F_X + \frac{7}{12} \cdot F_Y$

$EY = \int_0^1 x \frac{3}{7} dx + \int_2^3 x \frac{4}{7} dx = \left[\frac{3}{14} x^2\right]_0^1 + \left[\frac{4}{14} x^2\right]_2^3 =$ $\quad f_Y(x)$! $\quad (F_Y'$ DERIVACE$)$

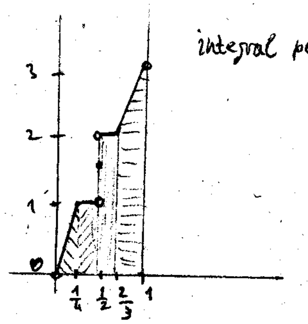
$= +\frac{3}{14} + \left(\frac{36}{14} - \frac{16}{14}\right) = \frac{23}{14}$

$EX = 1 \cdot \frac{3}{5} + 2 \cdot \frac{2}{5} = \frac{7}{5}$

$f(x) = \frac{dF}{dx}$

$EZ = \frac{5}{12} \cdot \frac{7}{5} + \frac{7}{12} \cdot \frac{23}{14} = \frac{7}{12} + \frac{23}{24} = \frac{37}{24}$

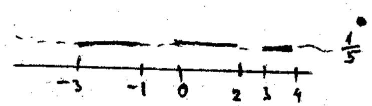
$EX = \int_0^1 Q(\alpha)\,d\alpha$ ... kvantilová funkce inverzní k distribuční



integrál plochy: $\frac{1}{8} + \frac{1}{4} + \frac{1}{6} \cdot 2 + \frac{1}{3} + \frac{1}{2} = \frac{37}{24}$

ROZPTYL

$DX = EX^2 - (EX)^2$ $\qquad X: f:$



$EX = \int_{-3}^{-1} x\frac{1}{5} dx + \int_0^2 x\frac{1}{5} dx + \int_3^4 x\frac{1}{5} dx = \left[\frac{x^2}{10}\right]_{-3}^{-1} + \left[\frac{x^2}{10}\right]_0^2 + \left[\frac{x^2}{10}\right]_3^4 = \frac{1}{10} - \frac{9}{10} + \frac{4}{10} + \frac{16}{10} - \frac{9}{10} = \frac{3}{10}$

$EX^2 = \int_{-3}^{-1} \frac{x^2}{5} dx + \int_0^2 \frac{x^2}{5} dx + \int_3^4 \frac{x^2}{5} dx =$

13.3.2006

$DX = E(X - EX^2) = E(X^2 - 2EX \cdot X + (EX)^2) = EX^2 - 2EX \cdot EX + (EX)^2 =$
$= EX^2 - (EX)^2$

Alternativní

$P[X=1] = p \in (0,1)$ $\qquad E(X) = \sum_{i=1}^{n} x_i P[X = x_i] = 0 \cdot (1-p) + 1p = p$
$P[X=0] = 1-p$

$D(X) = EX^2 - (EX)^2 = 0^2(1-p) + 1^2 p - p^2 = p - p^2 = p(1-p)$

Binomické

$P[X=k] = \binom{n}{k} p^k \cdot (1-p)^{n-k}$, $\quad k = 0, 1, \dots, n$

$EX = \sum_{k=0}^{n} k \cdot P[X=k] =$

Geometrické rozdělení

$P[X=k] = (1-p) p^k$, $\quad k = 0, 1, 2, \dots$

$\sum_{k=0}^{\infty} (1-p) \cdot p^k = (1-p) \sum_{k=0}^{\infty} p^k = (1-p) \cdot \frac{1}{1-p} = 1$ $\qquad \sum_{k=0}^{\infty} p^k = \frac{1}{1-p}$

$EX = \sum_{k=0}^{\infty} k(1-p) p^k = (1-p) \sum_{k=1}^{\infty} k p^k = (1-p) \cdot \frac{p}{(1-p)^2} = \frac{p}{1-p}$

$\frac{d}{dp} \sum_{k=1}^{\infty} k \cdot p^{k-1} = \frac{1}{(1-p)^2}$ $\quad / \cdot p$

$\sum_{k=1}^{\infty} k \cdot p^k = \frac{p}{(1-p)^2}$

$DX = EX^2 - (EX)^2 =$

$EX^2 = \sum_{k=0}^{\infty} k^2 (1-p) p^k = \frac{p(1+p)}{(1-p)^2}$

$\frac{d^2}{d^2p} \sum k^2 \cdot p^{k-1} = \frac{1+p}{(1-p)^3}$ $\quad / \cdot p$

$\sum_{k=1}^{\infty} k^2 \cdot p^k = \frac{p(1+p)}{(1-p)^3}$

Poissonovo rozdělení

$P[X=k] = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \qquad k = 0, 1, 2, \dots$

$e^{-\lambda} \sum \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$

$E(X) = e^{-\lambda} \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \lambda \sum_{k=1} \frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda$

$E(X^2) = e^{-\lambda} \sum_{k=1}^{\infty} k^2 \cdot \frac{\lambda}{k!} = e^{-\lambda} \cdot \lambda \left(\sum_{k=2}^{\infty} (k-1) \frac{\lambda^{k-1}}{(k-1)!} + e^{\lambda} \right) =$

$= e^{-\lambda} \cdot \lambda \left(\lambda \cdot \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} + e^{\lambda} \right) = e^{-\lambda} \cdot \lambda \left(\lambda \cdot e^{\lambda} + e^{\lambda} \right) =$

$= \lambda^2 + \lambda = \lambda(\lambda+1)$

$DX = \lambda(\lambda+1) - \lambda^2 = \lambda$

Rovnoměrné diskrétní

$P[X=k] = \frac{1}{n} \quad , \quad k = 1, 2, \dots, n$

$E(X) = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} k = \frac{1}{n} \cdot \frac{(n+1)n}{2} = \frac{n+1}{2}$

$E(X^2) = \sum_{i=1}^{n} k^2 \cdot \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} k^2$

$D(X) = \frac{1}{n} \sum_{i=1}^{n} k^2 - \left(\frac{n+1}{2} \right)^2$

Rovnoměrné spojité

$\frac{1}{b-a}$

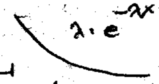
$EX = \int_{-\infty}^{\infty} x f(x)\, dx = \int_a^b x \frac{1}{b-a}\, dx = \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b = \frac{1}{b-a} \cdot \frac{b^2-a^2}{2} = \frac{a+b}{2}$

$EX^2 = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b = \frac{b^3-a^3}{3} \cdot \frac{1}{b-a} = \frac{b^2+ab+a^2}{3}$

$DX = -\left(\frac{a+b}{2} \right)^2 + \frac{b^2+ab+a^2}{3} = \frac{-3(a^2+2ab+b^2) + 4(b^2+ab+a^2)}{12} = \frac{(a-b)^2}{12}$

Exponenciální rozdělení

f



$\lambda \cdot e^{-\lambda x}$

$EX = \int_0^{+\infty} x \cdot \lambda e^{-\lambda x}\, dx = \frac{1}{\lambda} \int_0^{+\infty} \lambda t\, e^{-\lambda x}\, dt = + \frac{1}{\lambda} \cdot \Gamma(2) = \frac{1}{\lambda}$

$\int_0^{\infty} x^{a-1} \cdot e^{-x}\, dx = \Gamma(a)$

$EX^2 = \int_0^{\infty} x^2 \lambda \cdot e^{-\lambda x}\, dx = \lambda \Gamma(3) = 2 \cdot \frac{1}{\lambda^2}$

$\Gamma(1) = 1$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\Gamma(a+1) = a \cdot \Gamma(a)$

$DX = EX^2 - (EX)^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

20.3.2006

Normální rozdělení

$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \left| \begin{array}{l} x-\mu = t \\ dx = dt \end{array} \right| = \frac{1}{\sqrt{2\pi} \cdot \sigma} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}}\, dt =$

$= 2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^{\infty} e^{-\frac{t^2}{2\sigma^2}}\, dt = \left| \begin{array}{l} \frac{t^2}{2\sigma^2} = u \\ \frac{2t}{2\sigma^2} dt = du \end{array} \right| = \frac{2}{\sqrt{2\pi}\,\sigma} \int_0^{\infty} e^{-u} \cdot \frac{\sigma^2}{\sqrt{u 2\sigma^2}}\, du =$

$t = \sqrt{u 2\sigma^2}$

$dt = \frac{2\sigma^2}{2\sqrt{u 2\sigma^2}}$

$\Gamma(a) = \int_0^{\infty} x^{a-1} \cdot e^{-x}\, dx$

$= \frac{2\sigma^2}{\sigma\sqrt{4\pi\sigma^2}} \int_0^{\infty} u^{-\frac{1}{2}} e^{-u}\, du = \frac{\sigma}{\sigma\sqrt{\pi}} \cdot \Gamma(\tfrac{1}{2}) = 1$

$EX = \int_{-\infty}^{\infty} x \cdot F(x)\, dx = \mu$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t \cdot e^{-\frac{t^2}{2\sigma^2}}\, dt = K \int_0^{\infty} t \cdot e^{-\frac{t}{2\sigma^2}}\, dt = \Big| \qquad \Big| = K \cdot \int_0^{\infty} e^{-u} \cdot \sigma^2 du = k \cdot \sigma^2$

$EX^2 = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx = \sigma^2 + \mu^2$

$\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (t^2 + 2t\mu + \mu^2)\, e^{-\frac{t^2}{2\sigma^2}}\, dt = \dots$

$\to 0 \qquad \to \mu^2$

$= \frac{2}{\sqrt{2\pi}\,\sigma} \int_0^{\infty} t^2 e^{-\frac{t^2}{2\sigma}}\, dt = \Big| \qquad \Big| = k \cdot \int_0^{\infty} e^{-u} \sigma^2 \sqrt{u 2\sigma^2}\, du = \frac{2\sigma^2\sqrt{2\sigma^2}}{\sqrt{2\pi}\,\sigma} \int_0^{\infty}$

$$\Gamma\left(\frac{1}{2}\right) = 2\cdot\int_0^\infty \frac{1}{2} t^{-\frac{1}{2}} \cdot e^{-t}\,dt = \left| \begin{matrix} x = t^{\frac{1}{2}} \\ dx = \frac{1}{2t^{\frac{1}{2}}}\,dt \end{matrix} \right| = 2\int_0^\infty e^{-x^2}\,dx = 2\cdot I$$

$$* = \iint_{\substack{x>0\\y>0}} e^{-x^2-y^2}\,dx\,dy = \int_0^\infty \left(\int_0^\infty e^{-x^2}\,dx\right)\cdot e^{-y^2}\,dy = I\int_0^\infty e^{-y^2}\,dy = I^2 = \frac{\pi}{4}$$

$x = r\cdot\cos\varphi$ $\quad$ $y = r\cdot\sin\varphi$ $\qquad \frac{dx\,dy}{dr\,d\varphi} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cdot\cos\varphi \end{pmatrix} = r$ $\qquad I = \frac{\sqrt{\pi}}{2}$

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} 1\,d\varphi = \frac{1}{2}\int\left(\int e^{-s}\,ds\right)d\varphi = \frac{1}{2}\iint_{\substack{r>0\\ \varphi\in(0,\frac{\pi}{2})}} e^{-r^2}\, 2r\,dr\,d\varphi = *$$

$s>0$, $\varphi \in (0, \frac{\pi}{2})$

X, Y nezávislé

$$F_{X+Y}(t) = P[X+Y<t] = \begin{cases} X \text{ diskrétní } \sum_{x_i} P[X=x_i]\cdot P[x_i+Y<t] = \dots \\ X \text{ spojitá} \end{cases}$$

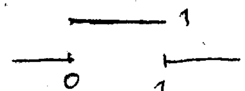
Y libovolná

$$\text{Diskrétní} \dots = \sum_{x_i} P[X=x_i]\cdot P[Y<t-x_i] = \sum_{x_i} P[X=x_i]\cdot F_Y(t-x_i)$$

$$\text{Spojitá} = \int_{\mathbb{R}} f_X(x)\cdot P[Y<t-x]\,dx = \int_{\mathbb{R}} f_X(x)\cdot F_Y(t-x)\,dx$$

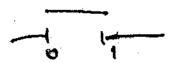
$$f_{X+Y}(t) = \frac{dF}{dt}(t) = \int_{\mathbb{R}} f_X(x)\,\frac{d}{dt}F_Y(t-x)\,dx = \int_{\mathbb{R}} f_X(x)\cdot f_Y(t-x)\,dx$$

X, Y nezávislé $\quad f$:

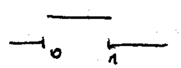


$$f_{X+Y}(t) = \int_0^1 1\cdot f(t-x)\,dx = \begin{cases} t\in(-\infty,0): \int_0^1 1\cdot 0\,dx = 0 \\ t\in(0,1): \int_0^t 1\,dx = t \\ t\in\langle 1,2): \int_{t-1}^{1} 1\,dx = 1-t+1 = 2-t \\ t\in\langle 2,\infty): \int_0 0\,dx = 0 \end{cases}$$

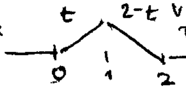
X:



Y:

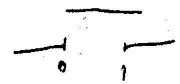


X+Y:

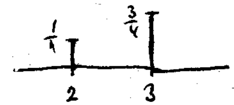


VZNIKÁ TROJÚHELNÍKOVÉ ROZDĚLENÍ

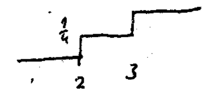
X:



Y:



F_Y:



$$F_{X+Y}(t) = \int_0^1 1\cdot F_Y(t-x)\,dx = \begin{cases} t\in(-\infty,2): 0 \\ t\in(2,3): \int_0^{t-2}\frac{1}{4}\,dt = \frac{t-2}{4} \\ t\in(3,4): \int_0^{t-3} 1\,dx + \int_{t-3}^{1}\frac{1}{4}\,dx = t-3-\frac{t-3}{4} = \frac{3(t-3)}{4} \\ t>4: 1 \end{cases}$$

27.3.2006

$Y = \sum_{i=1}^{n} X_i$, X_i Alt: $p = 0{,}02 = P[X_i = 1]$ $\quad n = 1000$

$P[Y\in\langle 10,30\rangle] = ?$ $\qquad EX_i = p \qquad DX_i = p(1-p)$

$EY = np \qquad DY = np(1-p)$

$EY = 20 \qquad DY = 20\cdot 0{,}98 = 19{,}6$

$(1000\cdot 2\cdot(1-0{,}02))$

1. ČN
2. CLV
3. Bi

1. Čebyševova nerovnost

$P[|X-EX|>\varepsilon] < \frac{DX}{\varepsilon^2}$ (musí existovat EX a DX)

$P[|X-EX|\le\varepsilon] \ge 1-\frac{DX}{\varepsilon^2}$

$P[|Y-EY|\le 10] \ge 1-\frac{19{,}6}{100} = 0{,}804$

2. $$P[Y\in\langle 10,30\rangle] = P[10\le Y\le 30] = P\left[\frac{10-EY}{\sqrt{DY}} \le \overbrace{\frac{Y-EY}{\sqrt{DY}}}^{Z} \le \frac{30-EY}{\sqrt{DY}}\right] = P\left[Z<\frac{30-EY}{\sqrt{DY}}\right] - P\left[Z\le\frac{10-EY}{\sqrt{DY}}\right] =$$

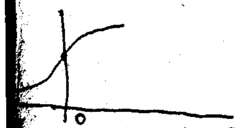
$\frac{Y-EY}{\sqrt{DY}} \sim N(0,1)$ $\qquad \Phi(t) = P[Z<t] = P[Z\le t]$

$P[a<Z<b] = P[Z<b] - P[Z\le a]$

$$= \Phi\left(\frac{30-20}{\sqrt{19{,}6}}\right) - \Phi\left(\frac{10-20}{\sqrt{19{,}6}}\right) = \Phi\left(\frac{10}{\sqrt{19{,}6}}\right) - \Phi\left(\frac{-10}{\sqrt{19{,}6}}\right) = 2\Phi\left(\frac{10}{\sqrt{19{,}6}}\right) - 1 =$$

$0{,}988083$

$$= 2\Phi(2{,}2588) - 1 = 0{,}976$$



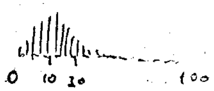
TABELACE POUZE KLADNÝCH HODNOT

$\Phi(-a) = 1-\Phi(a)$

$\Phi(a)-\Phi(-a) = \Phi(a) - 1 + \Phi(a) = 2\Phi(a) - 1$

3.

$P[Y=k] = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$, $k = 0,1,\dots,n$



$P[Y \in \langle 10,30 \rangle] = \sum_{k=10}^{30} \binom{n}{k} \cdot p^k (1-p)^{n-k} = \sum_{k=10}^{30} \binom{1000}{k} p^k (1-p)^{1000-k} =$

$= \sum_{k=10}^{30} \binom{1000}{k} \cdot 0{,}02^k (0{,}98)^{1000-k} \approx 0{,}982671$ ... přesné

Házíme kornou, kolikrát $n = ?$, ε je dáno.

$P[\bar{X} \in (\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)] \geq 0{,}95$

1. Čebyševova nerovnost

$P[|\bar{X} - E\bar{X}| \leq \varepsilon] \geq 0{,}95$

$E\bar{X} = \frac{1}{n} \sum EX_i = \frac{1}{2}$

$D\bar{X} = D(\frac{1}{n}\sum X_i) = \frac{1}{n^2} D(\sum X_i) = \frac{1}{n^2} \sum p(1-p) = \frac{1}{n} p(1-p) = \frac{1}{n} \cdot \frac{1}{4} = \frac{1}{4n}$

$= \frac{1}{n} DX_i$

$P[|\bar{X} - E\bar{X}| > \varepsilon] < \frac{D\bar{X}}{\varepsilon^2}$

$P[|\bar{X} - E\bar{X}| \leq \varepsilon] \geq (1 - \frac{D\bar{X}}{\varepsilon^2}) = \frac{\varepsilon^2 - \frac{1}{4n}}{\varepsilon^2} \geq 0{,}95$

$E\bar{X} = \frac{1}{2}$

$\varepsilon^2 - \frac{1}{4n} \geq 0{,}95 \cdot \varepsilon^2$

$\varepsilon^2 (1 - 0{,}95) \geq \frac{1}{4n}$

$\varepsilon^2 0{,}05 \geq \frac{1}{4n}$

$4n \geq \frac{1}{\varepsilon^2 0{,}05} = \frac{20}{\varepsilon^2}$, $n \geq \frac{5}{\varepsilon^2}$

ε	0,35	0,2	0,1	0,01
ČN	41	125	500	50 000

$P[\frac{1}{2} - \varepsilon < \bar{X} < \frac{1}{2} + \varepsilon] = P\left[\frac{\frac{1}{2} - \varepsilon - E\bar{X}}{\sqrt{D\bar{X}}} < \frac{\bar{X} - E\bar{X}}{\sqrt{D\bar{X}}} < \text{——}\right] = 2\Phi\left(\frac{\varepsilon}{\sqrt{D\bar{X}}}\right) - 1 = 0{,}95$

$\Phi\left(\frac{\varepsilon}{\sqrt{D\bar{X}}}\right) = \frac{1{,}95}{2} = 0{,}975$ $\quad \Phi\left(\frac{\varepsilon}{\frac{1}{2\sqrt{n}}}\right)$, $\frac{\varepsilon}{\frac{1}{2\sqrt{n}}} = 1{,}95996 = u$

$u_{0{,}975} = \Phi^{-1}(0{,}975)$

$\varepsilon = \frac{u}{2\sqrt{n}}$

$2 \cdot \varepsilon \cdot \sqrt{n} = u$

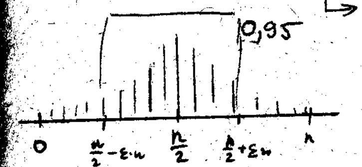
$\sqrt{n} = \frac{u}{2\varepsilon}$

$n = \left(\frac{u}{2\varepsilon}\right)^2$

ε	0,35	0,2	0,1	0,01
CLV	8	25	92	9604

$P[|\bar{X} - \frac{1}{2}| < \varepsilon] = P[|\sum X_i - \frac{n}{2}| < n \cdot \varepsilon] = P[n\varepsilon - \frac{n}{2} < B_i < n \cdot \varepsilon + \frac{n}{2}]$

$\sum X_i \rightarrow B_i$



ε	0,35	0,2	0,1	0,01
Bi	6	25	110	10 000

Výletní člun unese nosnost 5 tun, kolik cestujících může odvézt, aby pravděpodobnost přetížení byla $p = 0{,}001$

Váha 1 cestujícího $EX_i = 70$ kg , $\sqrt{DX_i} = 20$ kg.

$DX_i = 20^2$

centrální limitní věta.

$Y = \sum X_i$

$DY = \sum D_i = n \cdot 20^2$

$EY = E\sum X_i = n \cdot E$

$\frac{Y - EY}{\sqrt{DY}} = \frac{\sum X_i - n \cdot EX_i}{\sqrt{\sum DX_i}} = \frac{Y - n \cdot 70}{\sqrt{n} \cdot 20} \sim N(0,1)$

$P\Big[\underbrace{Z > \frac{5t - n70}{20\sqrt{n}}}_{\text{Normální normální rozdělení}}\Big] \leq 0{,}001$, $\Phi\left(\frac{5t - n70}{20\sqrt{n}}\right) \leq 0{,}999$

$\frac{5t - n70}{20\sqrt{n}} = u(0{,}999)$

$u_{0{,}999} = 3{,}09$

$3{,}09 = \frac{5t - n70}{20\sqrt{n}}$

$20\sqrt{n} \cdot u = 5t - n70$, $\left.\begin{matrix} s = \sqrt{n} \\ s^2 = n \end{matrix}\right\}$ substituce

$20 s u = 5t - s^2 70$

$70 s^2 + 20 s \cdot u - 5000 = 0$ $\qquad n \leq 64{,}34$

0.8.2.

x_i	0	1	2
p_i	p	q	$1-p-q$

... parametry p, q

3. 4. 2006

10 pokusů: 0 1 1 0 2 1 0 0 1 0

a) selským rozumem

$p = 0{,}5$

$q = 0{,}4$

b) 1. moment $m_1 = EX$ $\quad EX = 0 \cdot p + 1 \cdot q + 2(1-p-q) = -q + 2 - 2p$

2. moment $m_2 = EX^2$ $\quad EX^2 = 0^2 \cdot p + 1^2 q + 2^2(1-p-q) = -3q + 4 - 4p$

$m_1 = \frac{1}{n} \sum_i x_i = \frac{6}{10} = \frac{3}{5}$ $\qquad$ (1) $\frac{3}{5} = 2 - 2p - q$

$m_2 = \frac{1}{n} \sum_i x_i^2 = \frac{8}{10} = \frac{4}{5}$ $\qquad$ (2) $\frac{4}{5} = 4 - 4p - 3q$ $\Big\}$ $q = 0{,}4$, $p = 0{,}5$

c) metoda maximální věrohodnosti

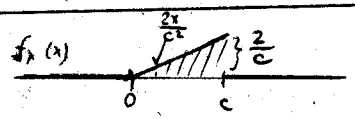
$$p^5 \cdot q^4 (1-p-q)^1 \overset{!}{=} MAX$$

$$f: \ln(p^5 \cdot q^4 \cdot (1-p-q)) = 5\ln p + 4 \ln q + \ln(1-p-q)$$

$$\frac{df}{dp} = \frac{5}{p} + \frac{-1}{1-p-q} = 0 \quad \Big\| \quad \frac{\partial f}{\partial q} = \frac{4}{q} + \frac{-1}{1-p-q} = 0$$

$$\begin{array}{l} 5-5p-5q-p=0 \\ 4-4p-4q-q=0 \end{array} \Bigg| \begin{array}{l} -6p-5q=-5 \\ -4p-5q=-4 \end{array} \Bigg) \quad \begin{array}{l} 2p=1 \\ p=0{,}5 \end{array} \quad q=0{,}4$$

X_{spoj}: $f_X(x)$



Data $x_1, x_2, \ldots, x_n$

1. Metoda momentů

$$m_1 = EX = \int_0^c x \cdot \frac{2x}{c^2}\,dx = \left[\frac{2x}{3c^2}\right]_0^c = \frac{2}{3}c \qquad \overline{x} = \frac{2}{3}c$$

$$c = \frac{3}{2}\overline{x}$$

2. $$L = \begin{cases} \frac{2x_1}{c^2} \cdot \frac{2x_2}{c^2} \cdot \ldots \cdot \frac{2x_n}{c^2} = MAX \\ 0, \quad \exists x_i \notin (0;c) \end{cases}$$ – věrohodnostní funkce hledá maximum

$\exists x_i > c$

$$c = \max(x_1, x_2, \ldots x_n)$$

Normální rozdělení

Data $x_1, x_2, \ldots, x_n$

$m_1 = EX \qquad EX = \mu$

$m_2 = EX^2 \qquad EX^2 = \sigma^2 + \mu^2$

$$m_1^2 = \frac{1}{n^2}\left(\sum_1^n x_i\right)^2 = \mu^2$$

$$m_2 = \frac{1}{n}\sum_1^n x_i^2 = \sigma^2 + \mu^2$$

$$\frac{1}{n}\sum_1^n x_i^2 - \overline{x}^2 = \sigma^2 = \frac{1}{n}\sum x_i^2 - \frac{1}{n^2}\left(\sum x_i\right)^2 = \frac{1}{n}\sum x_i^2 - \frac{1}{n^2}\left(\sum x_i^2 + 2\cdot\sum x_i x_j\right)$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \overline{x})^2 = \sum_{i=1}^n \left(x_i^2 - 2x_i\overline{x} + \overline{x}^2\right) = \sum_{i=1}^n x_i^2 - 2\sum_{i=1}^n x_i \frac{1}{2}\left(\sum_{j=1}^n x_j\right) +$$

$$+ n\left(\overline{x}\right)^2 = \sum_{i=1}^n x_i^2 - \frac{2}{n}\sum_i\sum_j x_i x_j + n\frac{1}{n^2}\left(\sum x_i\right)^2$$

$$f(x)_{N(\mu,\sigma^2)} = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\ell(\mu,\sigma^2) = \frac{1}{(\sqrt{2\pi}\,\sigma)^n}\prod_1^n e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = \frac{1}{(\sqrt{2\pi}\,\sigma)^n} e^{-\frac{1}{2\sigma^2}\sum_1^n (x_i-\mu)^2}$$

$$\ell'(\mu,\sigma^2) = -n \cdot \ln(\sqrt{2\pi}\cdot\sigma) - \frac{1}{2\sigma^2}\left[\sum_1^n (x_i-\mu)^2\right] \underbrace{\ln e}_{1} =$$

$$= -\frac{n}{2}\left[\ln 2\pi + \ln\sigma^2\right] - \frac{1}{2\sigma^2}\sum_1^n (x_i-\mu)^2$$

$$\frac{\partial \ell'}{\partial \mu} = -\frac{1}{2\sigma^2}\sum_1^n -2(x_i-\mu) = \frac{1}{\sigma^2}\sum_1^n (x_i-\mu) \overset{!}{=} 0$$

$$\frac{\partial \ell'}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \left(\frac{1}{\sigma^2}\right)^2 \sum_1^n (x_i-\mu)^2 \overset{!}{=} 0 \;\Big|\; (\sigma^2)^2$$

$$-\frac{n\sigma^2}{2} + \frac{1}{2}\sum_1^n (x_i-\mu)^2 = 0 \Rightarrow \sigma^2 = \frac{1}{n}\sum (x_i - \overline{x})^2 = m_2$$

$$= \frac{n-1}{n} S^2$$

$U, V, \quad U \sim Bi(2, \frac{1}{2})$

	0	1	2
U:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
V:	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

V: rovnoměrné 0, 1, 2

Momenty:

$$m_1 = c \cdot EU + (1-c)EV = c + 1 - c = 1$$

$$m_2 = EX^2 = 1^2 \cdot \frac{2+c}{6} + 4\,\frac{4-c}{12} = *$$

$X \sim Mix_c(U, V)$

data: 0 … μ_0, 1 … μ_1, 2 … μ_2 $\qquad c = ?$

$$c \cdot \frac{1}{4} + (1-c)\cdot\frac{1}{3} = \frac{c}{4} + \frac{1}{3} - \frac{c}{3} = \frac{1}{3} + \frac{3c-4c}{12} = \frac{4-c}{12}$$

$$* = \frac{2+c}{6} + \frac{4-c}{3} = \frac{2+c+8-2c}{6} = \frac{10-c}{6}$$

$$1 - \frac{4-c}{c} = \frac{2+c}{6}$$

10.8.4. Exponenciální rozdělení

$16, 0, 2, 0, 0,5, 1, 0, 2, 4$ (průměr $\frac{30}{10} = 3$)

$P = 0,95$

Mez ? $P[EX \in (-\infty, M)] = 0,95$

$L(\bar{x} + u_{1-\alpha} \cdot \frac{\sigma}{\sqrt{n}})$

$P[X < M] = 0,95$

$f(x) = \begin{cases} \lambda e^{-\lambda x} & (x > 0) \\ 0 & (x < 0) \end{cases} \rightarrow \frac{1}{3} e^{-\frac{1}{3}x}$

$\bar{X} = m_1 = EX$ $EX = \frac{1}{\lambda}$ $F_x(M) = 0,95$ ... distribuční funkce

$3 = \frac{1}{\lambda}$ $\lambda = \frac{1}{EX}$

$\lambda = \frac{1}{3}$

$F_x(M) = \int_0^t \frac{1}{3} e^{-\frac{1}{3}x} dx = \frac{1}{3}\left[-3e^{-\frac{1}{3}x}\right]_0^t =$

$= \frac{1}{3}\left[-3e^{-\frac{t}{3}} + 3\right] = 1 - e^{-\frac{t}{3}}$

$-e^{-\frac{1}{3}M} + 1 = 0,95$

$+e^{-\frac{1}{3}M} = +0,05$

$-\frac{1}{3}M = \ln 0,05$

$M = -3 \cdot \ln 0,05$, $M = 8,987$ ... do 8,9 minut přijede autobus do 0,95 %.

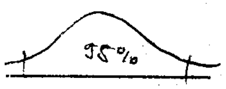
10.8.8. Při výrobě odporů, dochází k odchylce hodnoty $\sigma^2 = 0,8$

Vybereme 10 odporů: 9,91; 10; 10; 9,98; 10,02; 10,4; 9,82; 10,11; 10,17; 9,82

Předpokládá se normální rozdělení

Intervalový odhad EX pro 95% a 99%

$P[EX \in (\bar{x} - \delta, \bar{x} + \delta)] = 95\% = 1 - \alpha$
99 %



$P\left[EX \in \left(\bar{x} - u_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}, \bar{x} + u_{1+\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}\right)\right] = 1 - \alpha$

$t_{1-\frac{\alpha}{2}}(n-1) \cdot \frac{\sigma}{\sqrt{n}}$

$u_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = u_{1-0,025} \sqrt{\frac{0,8}{10}} = u_{0,975} \cdot \sqrt{0,08} = 1,95996 \cdot \sqrt{0,08} = 0,55$

$\bar{x} = \frac{\sum x_i}{10} = 10,023$

$P[EX \in (9,473; 10,573)] = 0,95$

$P[EX \in (9,2944; 10,752)] = 0,99$

10.8.7.

218; 219; 223; 219; 238 - napětí v síti, normální rozdělení

Najděte 95 % $\underset{\text{intervalový}}{\text{odhad}}$ EX a DX

$P\left[EX \in \left(\bar{x} - t_{1-\frac{\alpha}{2}}(n-1)\frac{S}{\sqrt{n}}, \bar{x} + t_{1-\frac{\alpha}{2}}(n-1)\frac{S}{\sqrt{n}}\right)\right] = 0,95$

$\bar{x} = \frac{1}{n}\sum x = 223,4$ $t_{0,975}(4) \cdot \frac{\sqrt{70,3}}{\sqrt{5}} = 2,7764 \cdot 3,7494 = 10,41$

$S^2 = \frac{1}{4}\left((5,4)^2 + (0,4)^2 + 4,4^2 + 4,4^2 + 4,6^2\right) = 70,3$

průměr mínus x_i

$P\left[DX \in \left(\frac{(n-1) \cdot S^2}{\chi^2_{1-\frac{\alpha}{2}}(n-1)}, \frac{(n-1) \cdot s^2}{\chi^2_{\frac{\alpha}{2}}(n-1)}\right)\right] = 1 - \alpha$

$\frac{4 \cdot 70,3}{\chi^2_{0,975}(4)} = \frac{281,2}{11,143} = 25,24$ $\frac{4 \cdot 70,3}{\chi^2_{0,025}(4)} = \frac{281,2}{0,48442} = 580,5$

$P = [DX \in (25,24; 580,5)] = 0,95$

10.8.6.

$n = 1000$, $p = 0,95$, najděte interval ve kterém bude s pravděpodobností $p = 0,95$ ležet součet bodů $= \sum x_i$

$P\left[\sum x_i \in (t_1, t_2)\right] = 0,95$

$0,95 = P[EX \in (\bar{x} - \delta, \bar{x} + \delta)] = P[\bar{x} - \delta < EX < \bar{x} - \delta] = P[\bar{x} < EX + \delta \wedge \bar{x} > EX - \delta] =$

$= P[EX - \delta < \bar{x} < EX + \delta] = P[\bar{x} \in (EX - \delta, EX + \delta)] = P[\sum x_i \in (nEX - n\delta, nEX + n\delta)]$

$EX = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = 3,5$

$DX = EX^2 - (EX)^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{91}{6} - \frac{49}{9} = 2,91$

$EX^2 = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$

$\delta = \underset{0,975}{u_{1-\frac{\alpha}{2}}} \frac{\sqrt{DX}}{\sqrt{n}} = 1,9596 \cdot \frac{\sqrt{2,91}}{\sqrt{1000}} = 0,105$

$P\left[\sum x_i \in (3500 - 105, 3500 + 105)\right]$ $(3395, 3605)$

10.5.17. 24.4.2006

X_n - posloupnost náhodných veličin vzájemně nezávislé

μ, σ^2

$n = ?$ že má platit, že pravděpodobnost $P[|\bar{X} - EX| \leq 0{,}1\sigma] \geq 0{,}95$

$P[-0{,}1\sigma \leq \bar{X} - EX \leq 0{,}1\sigma] \geq 1 - 0{,}05$

$P[EX \in (\bar{x} - \delta, \bar{x} + \delta)] = 1 - \alpha$

$\delta = u_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$

$P[\bar{x} + 0{,}1\sigma \geq EX \geq \bar{x} - 0{,}1\sigma] \geq 1 - 0{,}05$

$0{,}1\sigma = \delta = u_{1-\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$

$\sqrt{n} = u_{1-\frac{\alpha}{2}} \cdot 10$

$n = (u_{1-\frac{\alpha}{2}} \cdot 10)^2 = (19{,}5996)^2 \doteq 385$

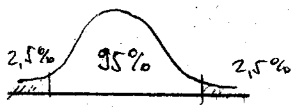
$u_{0{,}975} = 1{,}95996$ $\quad n > 385$

TESTOVÁNÍ HYPOTÉZ (intervalové odhady v jiném názvu)

$H_0: EX = \mu_0 \quad (EX \leq \mu_0)$

$P[EX \in (\bar{x} - \delta, \bar{x} + \delta)] = 1 - \alpha$

$P\left[\frac{EX - \bar{x}}{\sqrt{\frac{\sigma^2}{n}}} \in \left(-u_{1-\frac{\alpha}{2}}, u_{1-\frac{\alpha}{2}}\right)\right] = 1 - \alpha$



Zamítací pravidlo.

$H_1: EX \neq \mu_0 \quad (EX > \mu_0)$

10.9.1.

Deset balíčků cukru: 990 1010 985 950 975 975 990 1010 950 1010

můžeme na 95% rozhodnout hypotézu, že mají správnou váhu? $\bar{X} = 986$

$H_0: EX \geq 1000 \quad 1 - \alpha = 0{,}95$

$H_1: EX < 1000$

Rozptyl je známý $\sigma = 25$

$P[EX \in (\bar{x} - \delta, \bar{x} + \delta)]$

$P\left[\frac{-(EX - \bar{x})}{\sqrt{\frac{\sigma^2}{n}}} \in (-u_{1-\alpha}, \infty)\right] = 0{,}95$ ... nezamítáme H_0

0,99 ... zamítneme H_0

0,97 — HRANICE — p-hodnota testu

Zamítací pravidlo: $-\frac{(EX - \bar{x})}{\sqrt{\frac{\sigma^2}{n}}} < -u_{1-\alpha}$

$\frac{EX - \bar{x}}{\sqrt{\frac{\sigma^2}{n}}} > u_{1-\alpha}$ $\quad \frac{14 \cdot \sqrt{10}}{25} > 1{,}64485$

$1{,}77 > 1{,}64485$ - zamítáme H_0

$u_{0{,}99} = 2{,}32635$

$\underbrace{\phi\left(\frac{EX - \bar{x}}{\sqrt{\frac{\sigma^2}{n}}}\right)}_{\longrightarrow \text{ p-hodnota testu}} > 1 - \alpha$ $\quad \phi(1{,}77) = 0{,}9616$

Pokud $\sigma = ?$ - neznámé

$\sigma^2 = S^2$ - odhodneme

$u \rightarrow t(n-1)$

$\sigma^2 = s^2 \quad s^2 = \frac{1}{n-1} \sum_i (x_i - \bar{x})^2$

$s^2 = \frac{1}{9}(4^2 + 24^2 + 1^2 + 36^2 + 9^2 + \dots) = \frac{5540}{9} = 615{,}55$

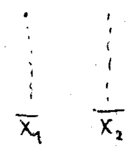
$\sqrt{s^2} = 24{,}8102$

Zamítací pravidlo: $\frac{EX - \bar{x}}{\sqrt{\frac{s^2}{n}}} > t_{1-\alpha}(n-1)$

$\frac{14 \cdot \sqrt{10}}{24{,}8102} > t_{0{,}95}(9)$ $\quad$ $0{,}99$

$1{,}78 > \begin{cases} t_{0{,}95}(9) = 1{,}8331 \\ t_{0{,}99}(9) = 2{,}8214 \end{cases}$ $\quad H_0$ nezamítáme

Dva nezávislé vzorky



$H_0: EX_1 = EX_2$

$H_1: EX_1 \neq EX_2$

-spárovat X_1 a X_2 Δ

$H_0: E(X_1 - X_2) = 0$

$\frac{|\bar{x}_1 - \bar{x}_2|}{s\sqrt{\frac{1}{n} + \frac{1}{m}}} > t_{1-\frac{\alpha}{2}}(n + m - 2)$

$s^2 = \frac{(m-1) \cdot s_1^2 + (n-1) s_2^2}{m + n - 2}$

$H_0': \sigma_1^2 = \sigma_2^2$

$H_1': \sigma_1^2 \neq \sigma_2^2$

větší $\rightarrow$ $\frac{s_1^2}{s_2^2} > F_{1-\frac{\alpha}{2}}(n-1, m-1)$ $\leftarrow$ menší

10.1.3.

Z_1 : 18, 19, 19, 21, 22, 20, 19, 21, ~~22~~ 22, 18, 19 | $\bar{X}$ 19,82

Z_2 : 19, 20, 21, 20, 18, 22, 21, 19, 21, 20, 21 | 20,18

$$S^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

$$S_1^2 = \frac{1}{10}\left(1{,}82^2 + 0{,}18^2 + \ldots\right) = 2{,}163$$

$$S_2^2 = \frac{1}{10}\left(1{,}18^2 + 0{,}18^2 + \ldots\right) = 1{,}363$$

$$S^2 = \frac{10 \cdot 2{,}163 + 10 \cdot 1{,}363}{20} = 1{,}763$$

$$\frac{|19{,}82 - 20{,}18|}{\sqrt{1{,}763} \cdot \sqrt{\frac{2}{11}}} > t_{0,975}(20)$$

$0{,}635 > 2{,}0860$ nezamítáme H_0

$$\frac{2{,}163}{1{,}363} > F_{0,975}(10,10), \quad 1{,}587 > 3{,}7168 \quad \text{nezamítáme } H_0$$

Znaménkový test

hypotéza $H_0 : \underset{\text{medián}}{\tilde{x}_{0,5}} = \check{c}$ $\quad P[X < \check{c}] = \frac{1}{2}$

Data – jsou větší nebo menší než $\check{c}$: + + + − − + − + + − − + + +

$Bi(n, \frac{1}{2})$

$H_1 : x_{0,5} \neq \check{c}$ – sestavíme zamítací pravidlo
počet + < kritická hodnota

Wilcoxonův test

$+c_1 + c_2 + c_3 - c_4 + c_5 + c_6 + c_7$

$+7 + 5 - 3{,}5 - 1 + 6 + 2 + 3{,}5$ – seřadíme podle velikosti

nejmenší

$\sum r_+ = 28{,}5$
$\sum r_- = 4{,}5$ $\Big\}$ min $4{,}5 \leq w(\alpha)_2$

$28 = 1 + 2 + \ldots + 7 = \frac{7 \cdot 8}{2} = 28$

10.9.6.

10 vzorků výrobku.

Před :	3,15	2,96	3,00	2,75	3,21	3,33	2,95	2,91	3,26	2,88
Po :	3,21	2,99	3,11	2,91	3,22	3,28	3,09	3,05	3,00	3,28

Liší se pevnost před a po zpracování?

Před – Po :	−0,06	−0,01	−0,11	−0,16	−0,01	+0,05	−0,14	−0,13	−0,02	−0,11
	5	1,5	6,5	9	1,5	4	8	10	3	6,5

$\sum r_+ = 4$
$\sum r_- = 51$ $\Big\}$ minimum $4 \leq w(\alpha)$
$\overline{55}$

~~w~~ $W_{10}(0{,}05) = 8$
$w_{10}(0{,}01) = 3$

$4 \leq w_{10}(0{,}01)$ nelze zamítnout
$4 \leq w_{10}(0{,}05)$ zamítáme

H_0 – data pochází z daného rozdělení (μ, λ)

$$\exp : f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

H_1 – ne

1957 – počty narozených dětí v jednotlivých měsících v ČSR

Měsíce :				
1.	21 182	31/365	21 465,6	3,75
2.	19 560	28/365	19 388,3	16,84
3.	22 787	31/365	21 465,6	81,34
4.	22 805	30/365	20 774,1	198,75
5.	23 120	31	21 465,6	127,57
6.	21 853	30		56,75
7.	21 367	31		0,45
8.	20 357	31		57,25
9.	20 946	30		1,44
10.	23 037	31		98,08
11.	18 708	30		201,33
12.	19 582	31		163,53

$$\frac{(21\,182 - 21\,465{,}6)^2}{21\,465{,}6}$$

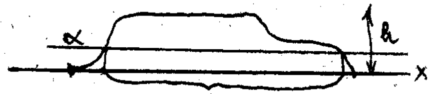
$n = 252\,740$

$1004{,}05 > 28{,}17$

Zamítáme H_0 (rodí se rovnoměrně)

FUZZY LOGIKA

9.5.2006



$h(A) = 1 \Rightarrow \text{core } A \neq 0$

α hladina $\alpha \in \langle 0,1 \rangle$

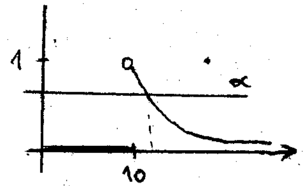
$R_A(\alpha) = \{x \in X,\ \mu_A(x) \geq \alpha\}$

$\alpha > \beta \Rightarrow R(\alpha) \subseteq R(\beta)$

$\bigcap_{\beta<\alpha} R(\beta) = R(\alpha)$

Př: $\mu_A(x) = \begin{cases} \frac{1}{1+(x-10)^2} & ;\ x > 10 \\ 0 & ;\ \text{jinde} \end{cases}$

Určete h, supp, core, horizontální reprezentaci



$h = 1$

$\text{supp} = (10, \infty)$

$\text{core} = \emptyset$ (tam kde je to množina jedničková)

$\frac{1}{1+(x+10)^2} = \alpha$

$1 = \alpha + \alpha(x-10)^2$, $x^2 - 20x + 101 = \frac{1}{\alpha}$

$x_{1,2} = \frac{20 \pm \sqrt{400 - 4\left(101 - \frac{1}{\alpha}\right)}}{2} = 10 \pm \sqrt{100 - 101 + \frac{1}{\alpha}} = 10 \pm \sqrt{\frac{1}{\alpha} - 1}$

$R(\alpha) = \left(10;\ 10 + \sqrt{\frac{1}{\alpha} - 1}\right)$

$R_A(\alpha) = \begin{cases} R & ;\ \alpha = 0 \\ \langle 0,\ 3-2\alpha \rangle & ;\ \alpha\ (0;\ \frac{1}{2}) \\ \langle 2(\alpha - \frac{1}{2})^2,\ 3-2\alpha \rangle & ;\ \alpha \in \langle \frac{1}{2}, 1) \end{cases}$

$x = 3 - 2\alpha$

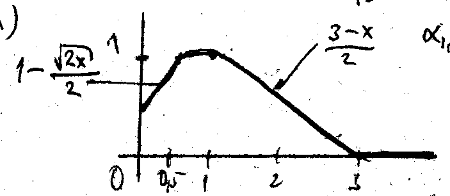
$\alpha = \frac{3-x}{2}$

$x = 2(\alpha - \frac{1}{2})^2$

$2\alpha^2 - 2\alpha + \frac{1}{2} - x = 0$

$\alpha_{1,2} = \frac{2 \pm \sqrt{2x}}{2}$

- vertikální reprezentace + graf $\mu(A)$



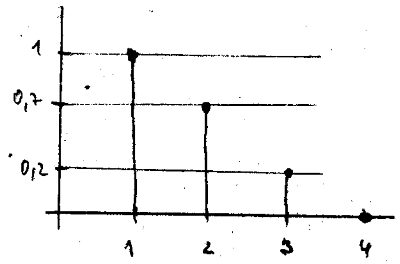
ad 1) ✓ $\text{supp}(A) = (0,3)$

ad 2) ✓ $\text{core}(A) = \langle 0{,}5;\ 1 \rangle$

ad 3) ✓

: Měření neporovnatelnosti, zkoumá se změna

$R_A(\alpha) = \begin{cases} \{1,2,3,4\} & \alpha = 0 \\ \{1,2,3\} & \alpha \in (0;\ 0{,}2\rangle \\ \{1,2\} & \alpha \in (0{,}2;\ 0{,}7) \\ \{1\} & \alpha \in (0{,}7;\ 1\rangle \end{cases}$



$A = \{(1;1), (2;0{,}7), (3;0{,}2), (4;0)\}$

Kardinalita $\text{Card } A = 1 + 0{,}7 + 0{,}2 + 0 = 1{,}9$

---

1. $\alpha > \beta \Rightarrow \neg\alpha \leq \neg\beta$
2. $\neg\neg\alpha = \alpha$

$\neg_s \alpha = 1 - \alpha$

$\neg : \langle 0,1 \rangle \to \langle 0,1 \rangle$

$i : \langle 0,1 \rangle \to \langle 0,1 \rangle$ rostoucí

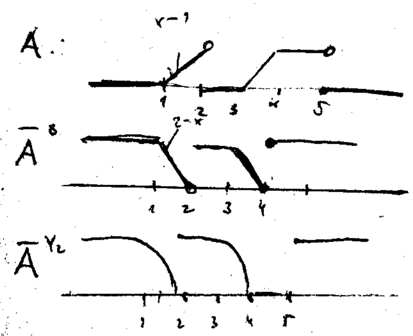
$\neg = i^{-1} \circ \neg_s \circ i$

$i = x^2$

$i = x^2$

~~$R_{\bar{A}}(x) = \dots$~~

$A \quad \bar{A} \qquad \mu_{\bar{A}}(x) = \neg \mu_A(x)$



$\bar{A}^s = ?$

$\bar{A}^{1/2} = ?$

15.5.

$\dot{\neg} : \langle 0,1\rangle \to \langle 0,1\rangle$

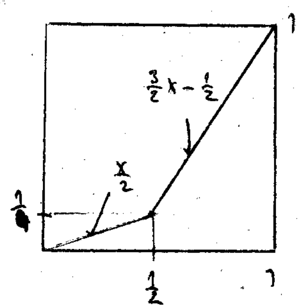
1) $\alpha \geq \beta \Rightarrow \neg\alpha \leq \neg\beta$

2) $\dot{\neg}\dot{\neg}\alpha = \alpha \qquad \neg_s \alpha = 1-\alpha$

i : rostoucí bijekce na $\langle 0,1\rangle$ $\quad \dot{\neg} = i^{-1}\left(\neg_s\, i(\alpha)\right)$

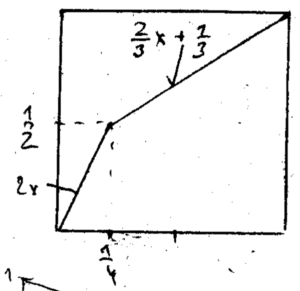
Př

i :



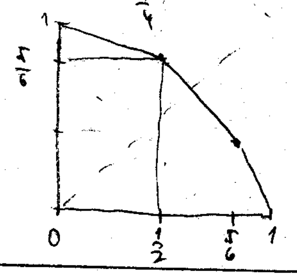
Najít negaci generovanou i

$$\dot{\neg}\alpha = \begin{cases} (\alpha \leq \frac{1}{2}) : i^{-1}\left(1-\frac{\alpha}{2}\right) * \\ (\alpha > \frac{1}{2}) : i^{-1}\left(1-\frac{3}{2}\alpha+\frac{1}{2}\right) \Delta \\ \qquad i^{-1}\left(\frac{3}{2}(1-\alpha)\right) \end{cases}$$



$* \;—\; \frac{2}{3}\left(1-\frac{\alpha}{2}\right)+\frac{1}{3} = 1$

$$\Delta \begin{cases} \left(\alpha \in \left(\frac{1}{2}, \frac{5}{6}\right\rangle\right) : \frac{2}{3} \\ \left(\alpha > \frac{5}{6}\right) : 2\,\frac{3}{2}(1-\alpha) = 3-3\alpha \end{cases}$$



2.98, 2.99

$\lambda \in (-1, \infty) \qquad \neg_s \alpha = \frac{1-\alpha}{1+\lambda\alpha}$

- je to negace?
- jakou bijekcí je generována?

1. $\alpha \geq \beta \quad \frac{1-\alpha}{1+\lambda\alpha} \overset{?}{\leq} \frac{1-\beta}{1+\lambda\beta} \qquad \beta \leq \alpha$

$(1+\lambda\beta)(1-\alpha) \leq (1-\lambda\alpha)(1-\beta)$

$\cancel{1} - \alpha + \lambda\beta - \cancel{\lambda\alpha\beta} \leq \cancel{1} - \beta + \lambda\alpha - \cancel{\lambda\alpha\beta}, \quad \lambda\beta + \beta \leq \lambda\alpha + \alpha$

$\beta(\lambda+1) \leq \alpha(\lambda+1)$

$$\neg_{s_\lambda}\neg_{s_\lambda}\alpha = \frac{1-\neg_{s_\lambda}\alpha}{1+\lambda\neg_s\alpha} = \frac{1-\frac{1-\alpha}{1+\lambda\alpha}}{1+\lambda\frac{1-\alpha}{1+\lambda\alpha}} = \frac{\frac{1+\lambda\alpha-1+\alpha}{1+\lambda\alpha}}{\frac{1+\lambda\alpha+\lambda-\lambda\alpha}{1+\lambda\alpha}} = \frac{\lambda\alpha+\alpha}{1+\lambda} = \frac{\alpha(\lambda+1)}{1+\lambda} = \alpha$$

$$i(\alpha) = \frac{\alpha+\neg_s\neg\alpha}{2} = \frac{\alpha+1-\frac{1-\alpha}{1+\lambda\alpha}}{2} = \frac{\alpha+\lambda\alpha^2+1+\lambda\alpha-1+\alpha}{2(1+\lambda\alpha)} = \frac{2\alpha+\lambda\alpha+\lambda\alpha^2}{2(1+\lambda\alpha)}$$

INVERZE

$\dot{\wedge}\dot{\vee} : \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$

1. $\alpha \dot{\wedge} \beta = \beta \dot{\wedge} \alpha$
2. $(\alpha \dot{\wedge} \beta)\dot{\wedge}\gamma = \alpha \dot{\wedge}(\beta \dot{\wedge} \gamma)$ $\Big\}$ stačí nahradit $\wedge$ za $\dot{\vee}$
3. $\beta \geq \gamma \Rightarrow \alpha \dot{\wedge} \beta \geq \alpha \dot{\wedge} \gamma$
4. $1 \dot{\wedge} \alpha = \alpha \mid 0 \dot{\vee} \alpha = \alpha$

$0 \dot{\wedge} \alpha \overset{(3)}{\leq} 0 \dot{\wedge} 1 \overset{(1)}{=} 1 \dot{\wedge} 0 \overset{(4)}{=} 0$

4 základní případy

$\alpha \wedge_s \beta = \min(\alpha,\beta)$	$\overset{\neg}{\longleftrightarrow}$	$\alpha \overset{s}{\vee} \beta = \max(\alpha,\beta)$	$\wedge\mid$
$\alpha \wedge_p \beta = \alpha \cdot \beta$	$\overset{\neg_s}{\longleftrightarrow}$	$\alpha \overset{p}{\vee} \beta = \alpha+\beta-\alpha\beta$	$\wedge\mid$
$\alpha \dot{\wedge}_L \beta = \alpha+\beta-1 \mid \langle 0,1\rangle$	$\overset{\neg_s}{\longleftrightarrow}$	$\alpha \overset{L}{\vee} \beta = \alpha+\beta \mid \langle 0,1\rangle$	$\wedge\mid$
$\alpha \wedge_D \beta = 0$ až na vlastnost 4	$\overset{\neg}{\longleftrightarrow}$	$\alpha \overset{D}{\vee} \beta = 1$ až na axiom 4	

- spojité
  - archimédovské: $\alpha \dot{\wedge} \alpha < \alpha$ na $(0,1)$, $\alpha \dot{\vee} \alpha > \alpha$ na $(0,1)$
    - striktní
    - nilpotentní
  - nearchimédovské
- nespojité

striktní: $\beta > \gamma \Rightarrow \alpha \dot{\wedge} \beta > \alpha \dot{\wedge} \gamma$ pro $\alpha \in (0,1\rangle \quad \overset{p}{\wedge} \quad \overset{p}{\vee}$

$\beta > \gamma \Rightarrow \alpha \dot{\vee} \beta > \alpha \dot{\vee} \gamma$ pro $\alpha \in \langle 0,1)$

nilpotentní: není striktní $\quad \underset{L}{\wedge} \quad \overset{L}{\vee} \qquad \alpha \dot{\wedge}\dot{\vee} \beta = i^{-1}\left(i(\alpha) \underset{p}{\wedge}\overset{p}{\vee} i(\beta)\right)$

i : bijekce na $\langle 0,1\rangle$ striktní

$\alpha \dot{\wedge}\dot{\vee} \beta = i^{-1}\left(i(\alpha) \underset{L}{\wedge}\overset{L}{\vee} i(\beta)\right)$

nilpotentní

Př.: $\alpha \overset{*}{\Delta} \beta = \dfrac{\alpha\beta}{2-\alpha-\beta+\alpha\beta}$

$$L = \frac{(\alpha\Delta\beta)\cdot\gamma}{2-\alpha\Delta\beta-\gamma+(\alpha\Delta\beta)\cdot\gamma} = \frac{\frac{\alpha\beta}{2-\alpha-\beta+\alpha\beta}\cdot\gamma}{2-\frac{\alpha\beta}{2-\alpha-\beta+\alpha\beta}-\gamma+\frac{\alpha\beta}{2-\alpha-\beta+\alpha\beta}\cdot\gamma} =$$

$$= \frac{\alpha\beta\gamma}{4-2\alpha-2\beta+2\alpha\beta-\alpha\beta-2\gamma+\alpha\gamma+\beta\gamma-\alpha\beta\gamma+\alpha\beta\gamma} =$$

$$= \frac{\alpha\beta\gamma}{4-2\alpha-2\beta-2\gamma+\alpha\beta+\alpha\gamma+\beta\gamma}$$

$$\frac{\partial}{\partial\beta}\Delta = \frac{\alpha(2-\alpha-\beta+\alpha\beta)-\alpha\beta\cdot(-1+\alpha)}{(\qquad)^2} \geq 0$$

$$2\alpha-\alpha^2-\alpha\beta+\alpha^2\beta+\alpha\beta-\alpha^2\beta$$

$\alpha(2-\alpha) > 0$ pro $\alpha \in (0,1\rangle$

$= 0$ pro $\alpha = 0$

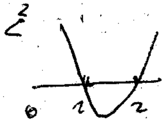
$$\frac{\beta}{2-1-\beta+\beta} = \frac{\beta}{1} = \beta$$

spojitá archimedovská

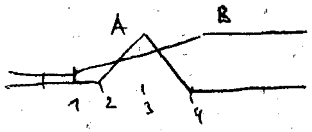
$$\alpha\Delta\alpha = \frac{\alpha^2}{2-2\alpha+\alpha^2} < \alpha$$

$\alpha < 2-2\alpha+\alpha^2$

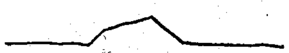
$0 < 2-3\alpha+\alpha^2$



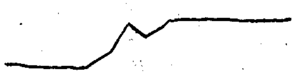
$A \cap B \quad \mu_A \wedge \vee \mu_B = \mu_{A \cap \cup B}$



std. průnik – minimum



std. sjednocení



kopří jak vypadají průniky

$A \underset{L}{\cap} B \quad A \underset{P}{\cap} B$

$A \underset{B}{\cap} B \quad A \overset{S}{\cup} B$

$A \overset{P}{\cup} B$

21.5. 2005

FUZZY RELACE

- $R \in \mathcal{F}(A\times B)$
  $R \in \mathcal{P}(A\times B)$

složení

$$R\circ S = \{(x,z);\ \exists y : (x,y) \in R \wedge (y,z) \in S\}$$

$$\mu_{R\circ S}(x,z) = \sup_{y\in B}\{\mu_R(x,y) \wedge \mu_S(y,z)\}.$$

$R \in \mathcal{F}(A\times B)$

$S \in \mathcal{F}(B\times C)$

Př.:

$A = B = \{0,1,2\}$

	0	1	2
0	0,1	0,2	0,3
1	0,4	0,5	0,6
2	0,7	0,8	0,9

maximum z minim 0,1

	0	1	2
0	0,5	0,5	0,5
1	0	0	1
2	1,0	0	0,7

$R \underset{S}{\circ} S$

$S \underset{S}{\circ} R$

$R \underset{L}{\circ} S$

$$\mu_{R\underset{S}{\circ}S}(0,1) = \sup_{y\in\{0,1,2\}}\{\mu_R(0,y) \underset{S}{\wedge} \mu_S(y,1)\} = \max\{\mu_R(0,0) \underset{S}{\wedge} \mu_S(0,1),\ \mu_R(0,1) \underset{S}{\wedge} \mu_S(1,1),\ \mu_R(0,2) \underset{S}{\wedge} \mu_S(2,1)\} =$$

$$= \max\{0{,}1 \underset{S}{\wedge} 0{,}5,\ 0{,}2 \underset{S}{\wedge} 0,\ 0{,}3 \underset{S}{\wedge} 0\} =$$

$$= \max\{0{,}1;\ 0;\ 0\} = 0{,}1$$

$R \underset{S}{\circ} S$:

	0	1	2
0	0,3	0,1	0,3
1	0,6	0,4	0,6
2	0,9	0,5	0,8

$S \underset{S}{\circ} R$

	0	1	2
0	0,5	0,5	0,5
1	0,7	0,8	0,9
2	0,7	0,7	0,7

Lukasiewiczova $\overset{\circ}{L}$ $\quad \alpha \underset{L}{\wedge} \beta = \begin{cases} \alpha+\beta-1 & \text{pro } \alpha+\beta-1 > 0, \\ 0 & \text{jinde.} \end{cases}$

$R \underset{L}{\circ} S$:

	0	1	2
0	0,1	0	0,2
1	0,6	0	0,5
2	0,9	0,2	0,5

Př.: $\mu_R(x,y) = \begin{cases} \frac{x+y}{2}, & x \in \langle 1; 2\rangle \ \ y \in \langle -1, 0\rangle \\ 0 & \text{jinde} \end{cases}$

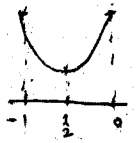
$R \in F(\mathbb{R}\times\mathbb{R})$:

$R \underset{s}{\circ} R, \ R \underset{p}{\circ} R$

$$\mu_{R \underset{s}{\circ} R}(x, z) = \sup_{y \in R} \{\mu_R(x,y) \underset{s}{\wedge} \mu_R(y,z)\} = \begin{cases} \left.\begin{matrix} x \in \langle 1,2\rangle \\ z \in \langle -1, 0\rangle \end{matrix}\right\} : 0 \\ \left.\begin{matrix} x \in \langle 1,2\rangle \\ \text{nebo} \\ z \notin \langle -1, 0\rangle \end{matrix}\right\} : 0 \end{cases}$$

$\mu_S(x,y) = \begin{cases} |y| \cdot \cancel{z} & \text{pro } y \in \langle -1, 1\rangle, \ z \in \langle 0, 1\rangle \\ 0 & \text{jinde} \end{cases}$

$R \underset{s}{\circ} S, \ R \underset{p}{\circ} S$



$$\mu_{R\circ S}(x,z) = \begin{cases} x \in \langle 1, 2\rangle \wedge z \in \langle 0,1\rangle \\ \sup\limits_{y \in \langle -1, 0\rangle} \{\frac{x+y}{2} \wedge |y| z\} \\ 0 \quad x \in \langle 1,2\rangle \vee z \in \langle 0, 1\rangle \end{cases}$$

$$\sup_{y \in \langle -1, 0\rangle} \left\{\frac{x+y}{2} \cdot -y z\right\} = \sup_{y \in \langle -1, 0\rangle} \left\{\frac{-xyz - y^2 z}{2}\right\}$$

$$\frac{df}{dy} = \frac{-xz}{2} - yz = 0 \quad y = -\frac{x}{2}$$

Reflexivita: $\mu_R(x,x) = 1$

Symetrie: $\mu_R(x,y) = \mu_R(y,z)$

Tranzitivita: $R \circ R \subseteq R$ $\quad \mu_R(x,y) \wedge \mu_R(y,z) \le \mu_R(x,z) \ \forall y$

$|x| \cdot y \wedge |y| \cdot z \le |x| \cdot z$